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RIEMANN–ROCH AND SMOOTHINGS OF SINGULARITIES

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INTRODUCTION

If M is a (not necessarily compact) complex n-manifold with the property that for some compact $K \subset M$ the subalgebra of $H^*(M-K)$ generated by the rational Chern classes is trivial in dim $\geq n$, then Chern numbers for M not involving the top class $c_n(M)$ may be defined almost as usual and—assuming that M has finite Betti numbers—$c_n[M]$ may be replaced by the euler characteristic $\chi(M)$. We encounter this situation when dealing with isolated singularities of complex spaces. Specifically, let X be a contractible Stein space of pure dimension n with isolated singular point $x \in X$ and let $\pi: \tilde{X} \to X$ be a resolution with exceptional set $E = \pi^{-1}(x)$. Then a deep result of O. Gabber implies that $H^*(\tilde{X}) \to H^*(\tilde{X} - E)$ is the zero map in dim $\geq n$ (if no coefficient groups are mentioned I mean rational cohomology). Hence $\tilde{X}$ admits Chern numbers. From this it is not hard to see that the Milnor fibre X_t of any smoothing of X (which after all can be given the same boundary as $\tilde{X}$) also admits Chern numbers. An interesting point is that, since $H^*(X_t)$ is trivial in dim $> n$, any nonzero Chern number of X_t is a universal linear combination of the euler characteristic and (if $n = 2m$ is even) c_m^2. The purpose of this paper is to relate these Chern numbers with analytic invariants of the smoothing or of the singularity itself. Our most general result in this direction is a Riemann–Roch defect theorem (3.3), whose proof depends on a globalization property of smoothings which we establish in an appendix. This globalization property (which was known to hold for certain classes of singularities, e.g. complete intersections and curve singularities) has been previously used for similar purposes by Milnor, Deligne, Laufer, and Wahl [14]. The fact that any smoothing can be globalized is especially relevant to this last paper, as many of its assertions need this as an extra hypothesis (see the remark at the end of this paper).

In practice it may be hard to use our results for singularities of dimension $n > 2$. Nevertheless we do have some applications to higher dimensional singularities. For instance, we prove that for any isolated singularity X pure *odd* dimension n, and smoothing component S of X, dim$(S) + 1/(n-1)!$ (euler characteristic of a general fibre over S) is independent of S. Furthermore, we find a new smoothability condition for odd dimensional X.

The Riemann–Roch defect relation is preceded by a signature defect relation (1.3). This is a slight generalization of a result of S. Morita, which has the virtue of applying to *all* isolated singularities of pure dimension, cf. (2.4). Moreover our proof is considerably simpler.

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§1. CHERN NUMBERS FOR CERTAIN NON-CLOSED MANIFOLDS

Let L be a compact closed $(2n-1)$-manifold with a complex structure J_L on $\tau_L \oplus \mathbb{R}_L$. We refer to such a pair (L, J_L) as an *almost contact* $(2n-1)$*-manifold*, but we suppress J_L if it is clear what it should be. We denote the rational Chern classes of this complex structure by $c_i(L, J_L)$ or $c_i(L)$. Suppose the following hypothesis verified.

(C) The subalgebra of $H^*(L)$ generated by the Chern classes of L is trivial in dim $\geq n$.

Since the complex cobordism groups are trivial in odd dimension [9], (L, J_L) bounds a compact weakly almost complex manifold M. (By this we mean that we are given a complex structure J_M on $\tau_M \oplus \mathbb{R}^{2k}_M$ for some k such that $\tau_M|\partial M$ is a complex subbundle, $\mathbb{R}^{2k}_{\partial M}$ has the standard complex structure ($J(e_{2i-1}) = e_{2i}$, $J(e_{2i}) = -e_{2i-1}$) and we are given a diffeomorphism $L \to \partial M$ such that its differential extends to a complex isomorphism $\tau_L \oplus \mathbb{R}_L \to \tau_M|\partial M$ making the unit section of $\mathbb{R}_L$ correspond to an outward pointing vector field along ∂M.) If $P = c_1^{i_1} \ldots c_n^{i_n}$ is any monomial in unknowns $c_1, \ldots, c_n$ of degree $2, \ldots, 2n$ respectively, then define $P[M] \in \mathbb{Q}$ as follows.

If deg $P \neq 2n$, set $P[M] = 0$.

If deg $P = 2n$ and P reducible: $P = P_1P_2$ with $n \leq \deg P_1 < 2n$, then by condition (C), $P_1(\tau_M) = P_1(c_1(\tau_M), \ldots, c_n(\tau_M))$ maps to zero in $H^*(\partial M) \simeq H^*(L)$. So $P_1(\tau_M)$ lifts to some $\tilde{P}_1(\tau_M) \in H^*(M, \partial M)$. Then $\tilde{P}_1(\tau_M)P_2(\tau_M) \in H^{2n}(M, \partial M)$ and we let $P[M]$ be the value of $\tilde{P}(\tau_M)P_2(\tau_M)$ on the fundamental class $[M] \in H_{2n}(M, \partial M)$ which is defined by the orientation induced by J_M. This definition is easily seen to be independent of the factorization of P.

In the remaining case: $P = c_n$, we proceed as follows. First observe that the homotopy class of an almost contact structure on L is nothing but a homotopy class $L \to BU_{n-1}$ whose composition with the natural map $BU_{n-1} \to BO_{2n-2} \to BO_{2n-1}$ classifies τ_L. Similarly, the weakly almost complex structure on M is a homotopy class $M \to BU$ lifting a classifying map $M \to BO$ for its stable tangent bundle. So we have a well-defined homotopy class of pairs $f: (M, \partial M) \to (BU, BU_{n-1})$. Since $H^*(BU_{n-1})$ resp. $H^*(BU)$ is the polynomial algebra on the universal Chern classes $c_1, \ldots, c_{n-1}$ resp. $c_1, c_2, \ldots$, it follows that $H^n(BU, BU_{n-1})$ may be identified with the $\mathbb{Q}$-span of c_n. We put $c_n[M] := \langle f^*c_n, [M] \rangle$. We will often use the following fact.

LEMMA 1.1. *If M is almost complex, then $c_n[M] = \chi(M)$.*

Proof. In that case f factors over f': $(M, L) \to (BU_n, BU_{n-1})$. The universal bundle γ^n restricted to BU_{n-1} splits as $\gamma^{n-1} \oplus \mathbb{C}_{BU_{n-1}}$ and hence admits a canonical section. The relative Chern class $c_n \in H^{2n}(BU_n, BU_{n-1})$ is just the obstruction class to extending this section over BU_n. So $f'^*(c_n)$ is the obstruction to extending an inward pointing vector field along ∂M to M. Evaluated on $[M]$ this gives of course $\chi(M)$.

We extend the definition of $P[M]$ to all $P \in \mathbb{Q}[c_1, \ldots, c_n]$ by linearity. It is clear that we recover the usual definition if $\partial M = \varnothing$.

Within the homotopy class of J_L we can find a complex structure which commutes with the involution $i = 1 \oplus -1$ on $\tau_L \oplus \mathbb{R}_L$. Denoting this complex structure still by J_L, a new complex structure on $\tau_L \oplus \mathbb{R}_L$ is given by $J_L \circ i$. Its homotopy class only depends on that of J_L. The resulting almost contact manifold is denoted $-(L, J_L)$ or simply $-L$. If L resp. $-L$ bound compact weakly almost complex manifolds M resp. M', then we can glue M and M' along L and get a closed compact weakly almost complex manifold N. In this situation it is easily seen that

1.2. $$P[N] = P[M] + P[M']$$

for all $P \in \mathbb{Q}[c_1, \ldots, c_n]$.

As an application of this simple fact, suppose $n = 2m$ even and let us for P take the Hirzebruch L-polynomial $L_m \in \mathbb{Q}[c_1, \ldots, c_n]$ as a polynomial in the Chern classes (e.g.

$L_1 = 1/3(c_1 - 2c_2)$). The following proposition strengthens a result of Morita [10].

PROPOSITION 1.3. *Given L as above, then for any compact weakly almost complex manifold M admitting L as boundary, the signature defect sign $(M) - L_m[M] \in \mathbb{Q}$ only depends on the almost contact manifold L.*

Proof. Let M' be a compact weakly complex manifold admitting $-L$ as boundary and let N be obtained by gluing M and M' along L. Then the Hirzebruch signature theorem asserts that $L_m[N] = \operatorname{sign}(N)$. The signature is additive: $\operatorname{sign}(N) = \operatorname{sign}(M) + \operatorname{sign}(M')$ and so is L_m: $L_m[N] = L_m[M] + L_m[M']$ (by (1.2a)). It follows that M and M' have opposite signature defect. So if L also bounds the compact weakly almost complex manifold M_1, then M and M' have equal signature defect.

Remark 1.4. When dealing with an open weakly almost complex $2n$-manifold M, the following substitute for condition (C) is convenient.

(C′) There exists a compact subset $K \subset M$ such that subalgebra of $H^*(M-K)$ generated by the Chern classes vanishes in degree $\geq n$.

In that case Chern numbers $P[M]$ not involving c_n ($P \in \mathbb{Q}[c_1, \ldots, c_{n-1}]$) can be defined in a similar fashion: if $P_1 \in \mathbb{Q}[c_1, \ldots, c_{n-1}]$ is of degree $\geq n$, $< 2n$, then $P_1(\tau_M)$ lifts to $\tilde{P}_1(\tau_M)P_2(\tau_M) \in H^{2n}(M)$ can be evaluated on the fundamental cycle. We have the following analogue of (1.2a): if K splits into disjoint compact K_1, K_2, put $M_1 := M - K_2$, $M_2 := M - K_1$. Then

$$P[M] = P[M_1] + P[M_2] \tag{1.4a}$$

for all $P \in \mathbb{Q}[c_1, \ldots, c_{n-1}]$. If $M - K$ has finite Betti numbers then so has M and, we can define $c_n[M] := \chi(M)$. But (1.4a) will not hold for c_n unless $\chi(M-K) = 0$.

§2. CHERN NUMBERS FOR RESOLUTIONS AND MILNOR FIBRES

By X we denote a Stein space of pure dimension n, having a unique singular point $x \in X$. We suppose that X is a good Stein representative of the germ (X, x) (so X can be given a smooth boundary L with $\bar{X} := X \cup L$ homeomorphic to the cone over L). We call L a *link* of (X, x).) By a *resolution* of X we will always mean a proper morphism $\pi\colon \tilde{X} \to X$ with nonsingular $\tilde{X}$ which is an isomorphism over $X - \{x\}$.

LEMMA 2.1. *The link L satisfies condition (C), or equivalently, for any resolution $\pi\colon \tilde{X} \to X$ of X, $\tilde{X}$ satisfies condition (C′).*

Proof. Let $\pi\colon \tilde{X} \to X$ be a resolution and $E := \pi^{-1}(x)$ its exceptional set. A corollary of a deep theorem due to Gabber is that the restriction map $H^*(\tilde{X}) \to H^*(\tilde{X} - E)$ is trivial in degree $\geq n$ (see Steenbrink [13] for a proof using mixed Hodge theory).

2.2. Recall that a *smoothing* of the germ (X, x) over $(\mathbb{C}, 0)$ consists of a flat morphism $f\colon (\mathscr{X}, x_0) \to (\mathbb{C}, 0)$ of germs of complex-analytic spaces together with an isomorphism $(X, x) \cong (f^{-1}(0), x_0)$, such that the general fibre of f is smooth. We shall usually identify (X, x) with $(f^{-1}(0), x_0)$. Given such a smoothing we can find a good representative $f\colon \mathscr{X} \to \Delta$ (where Δ is an open disc about $0 \in \mathbb{C}$) with the following properties: $\mathscr{X}$ is Stein, f is smooth outside $\{x\}$, there exists a neighbourhood K of x, proper over Δ, such that $f\colon \mathscr{X} - K \to \Delta$ is topologically trivial and $X := f^{-1}(0)$ is a good representative of $(f^{-1}(0), x)$.

LEMMA 2.3. *Any fibre X_t with $t \neq 0$ satisfies condition (C′). Its only possible nonzero Chern numbers are $\mathbb{Q}$-linear combinations of χ and, if $n = 2m$ is even, c_m^2.*

Proof. For all $t \in \Delta$, the inclusion $X_t - K_t \subset \mathscr{X} - K$ induces an isomorphism on cohomology taking Chern classes to Chern classes. Since $X_0 - K_0 \cong \tilde{X} - \pi^{-1}(K_0)$ satisfies condition (C), so does $X_t - K_t$. Hence for $t \neq 0$, X_t satisfies (C′). Since X_t is Stein, its cohomology in dim $> n$ is trivial so that any nonzero Chern number is a linear combination of χ and $c_{1/2n}^2$.

COROLLARY 2.4. *For $n = 2m$, even we have* $\operatorname{sign}(X_t) - L_m[X_t] = \operatorname{sign}(\tilde{X}) - L_m[\tilde{X}]$.

§3. THE RIEMANN-ROCH DEFECT

3.1. Suppose that for any flat family $g\colon \mathscr{X} \to S$ of complex spaces of dimension n we are given a coherent $\mathcal{O}_{\mathscr{X}}$-module $\mathscr{F}_{\mathscr{X}/S}$ which is natural in the sense that

(i) If $\mathscr{X}' \subset \mathscr{X}$ is open, then we are given a functorial $\mathcal{O}_{\mathscr{X}'}$-isomorphism $\mathscr{F}_{\mathscr{X}/S}|\mathscr{X}' \to \mathscr{F}_{\mathscr{X}'/S}$

(ii) Given a cartesian square

$$\begin{array}{ccc} \mathscr{X}' & \xrightarrow{h} & \mathscr{X} \\ {\scriptstyle g'}\downarrow & & \downarrow{\scriptstyle g} \\ S' & \rightarrow & S \end{array}$$

then there is a functorial $\mathcal{O}_{\mathscr{X}}$-homomorphism $h^*\mathscr{F}_{\mathscr{X}/S} \to \mathscr{F}_{\mathscr{X}'/S'}$ which is an isomorphism where h is smooth.

(iii) If Z is a complex analytic space of dimension n with zero-dimensional singular locus Σ and $\pi\colon \tilde{Z} \to Z$ is a resolution of Z, then there exists a coherent ideal $\mathscr{I} \subset \mathcal{O}_Z$ which defines Σ and is such that the isomorphism $\mathscr{F}_{Z-\Sigma} \to \pi_*\mathscr{F}_{\tilde{Z}}|Z - \Sigma$ extends to $\mathscr{I}\mathscr{F}_Z \to \pi_*\mathscr{F}_{\tilde{Z}}$.

Examples of such sheaves are $\mathcal{O}_{\mathscr{X}}$, $\Omega_{\mathscr{X}/S}$, $\Theta_{\mathscr{X}/S}$ etc. Observe that if $g\colon \mathscr{X} \to S$ is smooth, then $\mathscr{F}_{\mathscr{X}/S}$ is locally free. For $\mathscr{F}_{\mathbb{C}^{n+k}/\mathbb{C}^k}$ is locally free in some point p_k of $\mathbb{C}^{n+k}$, and for every $z \in \mathscr{X}$, the germs of g at z and of $\mathbb{C}^{n+k} \to \mathbb{C}^k$ at p_k fit in a cartesian diagram as above. In particular, $\mathscr{F}_{\mathbb{C}^n,0}$ is a free $\mathcal{O}_{\mathbb{C}^n,0}$-module. The functoriality properties imply that the group of automorphisms of $\mathbb{C}^n, 0$ acts in a natural manner on $\mathscr{F}_{\mathbb{C}^n,0}$. The first order part of this action of this action yields a representation ρ of $GL_n\mathbb{C}$ on the fibre $F := \mathscr{F}_{\mathbb{C}^n,0}/m_{\mathbb{C}^n,0}\mathscr{F}_{\mathbb{C}^n,0}$. Now if Z is a complex analytic manifold of complex dimension n, then a standard argument from the theory of fibre bundles gives that the complex vector bundle over Z defined by $\mathscr{F}_Z$ is isomorphic to $\mathscr{GL}_Z \times_{GL_n\mathbb{C}} F$ (we disregard the analytic structures on these bundles), where $\mathscr{GL}_Z \to Z$ denotes the principal $GL_n\mathbb{C}$-bundle associated to the tangent bundle of Z. This implies that there is a universal polynomial $C_{\mathscr{F},i} \in \mathbb{Q}[c_1, \ldots, c_n]$ (which only depends on ρ) such that its value on the Chern classes of any such Z gives the ith Chern class of $\mathscr{F}_Z$. If moreover Z is compact, then the coherent cohomology groups of $\mathscr{F}_Z$ are finite dimensional and vanish in $\dim \geq n$, so that the Euler–Poincaré characteristic $\chi(\mathscr{F}_Z) := \Sigma_{i \geq 0} (-1)^i \dim_{\mathbb{C}} H^i(\mathscr{F}_Z)$ is defined. The Riemann–Roch theorem (as generalized by Atiyah–Singer) asserts that $\chi(\mathscr{F}_Z)$ is given by a Chern number of Z. More specifically,

3.1a. $$\chi(\mathscr{F}_Z) = (\mathrm{ch}(C_{\mathscr{F},1}, \ldots, C_{\mathscr{F},n}) \cdot T_n)[Z],$$

where $T_n \in \mathbb{Q}[c_1, \ldots, c_n]$ is the nth Todd polynomial (see [7]) and ch denotes the Chern character. Let $P_{\mathscr{F}}$ denote the degree $2n$-part of the polynomial occurring in the right hand side of (3.1a), so that (3.1a) becomes

3.1b. $$\chi(\mathscr{F}_Z) = P_{\mathscr{F}}[Z].$$

3.2. Now let $f\colon \mathscr{X} \to \Delta$ be a good representative of a smoothing of an isolated singularity $X \ni x$ of pure dimension n (as in the previous section) and let $\pi\colon \tilde{X} \to X$ be a resolution of the central fibre $X = X_0$. Then conditions (i) and (ii) imply that we have a natural $\mathcal{O}_{X-\{x\}}$-homomorphism

$$\alpha\colon (\mathcal{O}_X \otimes \mathscr{F}_{\mathscr{X}/\Delta})|X - \{x\} \to \pi_* \mathscr{F}_{\tilde{X}}|X - \{x\}$$

and according to (iii), there exists a coherent ideal $\mathscr{I} \subset \mathcal{O}_X$ which defines x and is such that α extends to a $\mathcal{O}_X$-homomorphism

$$\alpha_{\mathscr{I}}\colon \mathscr{I} \otimes \mathscr{F}_{\mathscr{X}/\Delta} \to \pi_* \mathscr{F}_{\tilde{X}}.$$

Let $\beta_{\mathscr{I}}\colon \mathscr{I} \otimes \mathscr{F}_{\mathscr{X}/\Delta} \to \mathcal{O}_X \otimes \mathscr{F}_{\mathscr{X}/\Delta}$ be the obvious homomorphism. Clearly kernel and cokernel of both $\alpha_{\mathscr{I}}$ and $\beta_{\mathscr{I}}$ are supported by $\{x\}$ and hence finite dimensional. So for both homomorphisms the index (= dim kernel dim cokernel) is defined. We claim that index $(\alpha_{\mathscr{I}})$ —index $(\beta_{\mathscr{I}})$ is independent of $\mathscr{I}$. This will follow from the following simple and well-known lemma whose proof is left to the reader.

LEMMA. *Let $u\colon A_1 \to A_2$ and $\alpha_i\colon A_i \to B$ $(i = 1, 2)$ be homomorphisms of complex vector spaces such that $\alpha_1 = \alpha_2 u$. Then we have a natural exact sequence* $0 \to \operatorname{Ker}(u) \to \operatorname{Ker}(\alpha_1) \to \operatorname{Ker}(\alpha_2) \to \operatorname{Coker}(u) \to \operatorname{Coker}(\alpha_1) \to \operatorname{Coker}(\alpha_2) \to 0$. *In particular, if u and α_2 are Fredholm maps, then so is α_1 and we have that* $\operatorname{index}(\alpha_1) = \operatorname{index}(\alpha_2) + \operatorname{index}(u)$.

To prove our claim, let $\mathscr{J} \subset \mathscr{I}$ be a coherent ideal which defines $\{x\}$ and denote by $u\colon \mathscr{J} \otimes \mathscr{F}_{\mathscr{X}/\Delta} \to \mathscr{I} \otimes \mathscr{F}_{\mathscr{X}/\Delta}$ be the natural homomorphism. Applying the lemma to both $(u, \alpha_{\mathscr{J}}, \alpha_{\mathscr{I}})$ and $(u, \beta_{\mathscr{J}}, \beta_{\mathscr{I}})$ gives that the difference of the index of $\alpha_{\mathscr{J}}$ and of $\beta_{\mathscr{J}}$ resp. of $\alpha_{\mathscr{I}}$ and of $\beta_{\mathscr{I}}$ is the same. Since the ideals defining $\{x\}$ form an inverse system, this proves our claim.

We now define the *$\mathscr{F}$-index* of the diagram $X \subset \mathscr{X} \to \Delta$ by $\operatorname{index}(\alpha_{\mathscr{I}}) - \operatorname{index}(\beta_{\mathscr{I}}) - \dim(\text{torsion of } \mathcal{O}_X \otimes \mathscr{F}_{\mathscr{X}/\Delta})$ and denote it occasionally by index (α). Our main result is a formula for the $\mathscr{F}$-index:

THEOREM 3.3. *Let X_t be a nonsingular fibre of $f\colon \mathscr{X} \to \Delta$. Then*

$$\operatorname{index}(\alpha) = p_{\mathscr{F}}[X_t] - p_{\mathscr{F}}[X] + \Sigma_{i \geq 1}(-1)^i h^i(\mathscr{F}_{\tilde{X}}).$$

Proof. Following the appendix we can assume that f is the restriction of a projective flat family $F\colon \mathscr{Z} \to \Delta$ which is smooth outside x. We write Z for Z_0 $(= F^{-1}(0))$. Notice that the resolution $\pi\colon \tilde{X} \to X$ and the identity map of $Z - \{x\}$ can be glued to form a resolution $\Pi\colon \tilde{Z} \to Z$ of Z.

Put $\bar{\mathscr{F}}_{\mathscr{Z}/\Delta} = \mathscr{F}_{\mathscr{Z}/\Delta}/\text{torsion}$. Then $\bar{\mathscr{F}}_{\mathscr{Z}/\Delta}$ is flat over Δ. Combining this with the fact that $\bar{\mathscr{F}}_{\mathscr{Z}/\Delta}|\mathscr{Z} - \{x\}$ is locally free, it follows that $\mathcal{O}_Z \otimes \bar{\mathscr{F}}_{\mathscr{Z}/\Delta}$ is torsion free. So $\mathcal{O}_Z \otimes \bar{\mathscr{F}}_{\mathscr{Z}/\Delta} \simeq (\mathcal{O}_Z \otimes \mathscr{F}_{\mathscr{Z}/\Delta})/\text{torsion}$. According to a fundamental result of Grothendieck [11], $\chi(\mathcal{O}_{Z_s} \otimes \bar{\mathscr{F}}_{\mathscr{Z}/\Delta})$ is independent of $s \in \Delta$. Hence

(a) $$\begin{aligned}\chi(\mathscr{F}_{Z_t}) &= \chi(\mathcal{O}_{Z_t} \otimes \bar{\mathscr{F}}_{\mathscr{Z}/\Delta}) \\ &= \chi(\mathcal{O}_Z \otimes \bar{\mathscr{F}}_{\mathscr{Z}/\Delta})\end{aligned}$$

$$= \chi(\mathcal{O}_Z \otimes \mathcal{F}_{\mathcal{X}/\Delta}) - \dim(\text{torsion of } \mathcal{O}_{X,x} \otimes \mathcal{F}_{\mathcal{X}/\Delta})$$
$$= \chi(\Pi_* \mathcal{F}_{\tilde{Z}}) + \text{index}(\alpha).$$

The Leray spectral sequence for Π yields

(b) $$\chi(\mathcal{F}_{\tilde{Z}}) = \chi(\Pi_* \mathcal{F}_{\tilde{Z}}) + \Sigma_{i \geq 1} (-1)^i \dim_{\mathbb{C}} (R^i \Pi_* \mathcal{F}_{\tilde{Z}})_x,$$

and since X is Stein, we may replace $(R^i \Pi_* \mathcal{F}_{\tilde{Z}})_x$ by $H^i(\mathcal{F}_{\tilde{X}})$. By Riemann–Roch (3.1b) and the additivity of Chern numbers (1.2):

(c) $$\chi(\mathcal{F}_{Z_t}) = p_{\mathcal{F}}[Z_t] = p_{\mathcal{F}}[X_t] + p_{\mathcal{F}}[Z_t - X_t] \text{ and}$$
$$\chi(\mathcal{F}_{\tilde{Z}}) = p_{\mathcal{F}}[\tilde{Z}] = p_{\mathcal{F}}[\tilde{X}] + p_{\mathcal{F}}[Z - X].$$

Since $Z_s - X_s \subset \mathcal{Z} - \mathcal{X}$ induces an isomorphism on cohomology which takes Chern classes to Chern classes (for all $s \in \Delta$), $p_{\mathcal{F}}[Z_t - X_t] = P_{\mathcal{F}}[Z - X]$. The theorem is now a direct consequence of this and the formulas (a)–(c).

§4. APPLICATIONS AND EXAMPLES

4.1. In this section X denotes a reduced isolated singularity of pure dimension $n \geq 1$ and $f: \mathcal{X} \to \Delta$ is a good representative of a smoothing of X. If $\pi: \tilde{X} \to X$ is a resolution (which as usual is supposed to be an isomorphism over $X - \{x\}$), then $\pi_* \mathcal{O}_{\tilde{X}}$ can be identified with the normalization $\tilde{\mathcal{O}}_X$ of $\mathcal{O}_X$. The *arithmetic genus* $p(X)$ of (X, x) is defined by

$$(-1)^n p(X) = \Sigma_{i \geq 1} (-1)^{i-1} h^i(\mathcal{O}_{\tilde{X}}) - \dim_{\mathbb{C}} \pi_* \mathcal{O}_{\tilde{X}} / \mathcal{O}_X.$$

This number is independent of the resolution and thus constitutes an invariant of X. Applying our theorem to $\mathcal{F} = \mathcal{O}$, we find

4.1a. $$T_n[X_t] = T_n[\tilde{X}] + (-1)^n p(X).$$

For odd n, T_n is divisible by c_1 [7], so that

4.1b. $$T_n[X_t] = \begin{cases} \frac{1}{2}\chi(X_t) & \text{if } n = 1 \\ 0 & \text{if } n \text{ is odd} \geq 3 \\ \alpha_m \chi(X_t) + \beta_m c_m^2(X_t) & \text{if } n = 2m, \end{cases}$$

where α_m resp. β_m denote the coefficients of c_{2m} and c_m^2 respectively in the Todd polynomial T_{2m}. For $n = 1$, (a) yields Milnor's formula:

$$\chi(X_t) = \#(\text{irred. comp. of } X) - 2\dim_{\mathbb{C}} \tilde{\mathcal{O}}_X / \mathcal{O}_X.$$

For $n = 2$, we get

4.1c. $$\frac{1}{12}(\chi(X_t) + c_1^2(X_t)) = \frac{1}{12}(\chi(\tilde{X}) + c_1^2(\tilde{X})) + p(X)$$

Morita's result (1.3) gives

4.1d. $$\frac{1}{3}(-2\chi(X_t) + c_1^2(X_t)) - \text{sign } X_t = \frac{1}{3}(-2\chi(\tilde{X}) + c_1^2(\tilde{X})) - \text{sign } \tilde{X}$$

Combining these two results gives after a little bit of work (see [8]) that

4.1e. $$\mu_0 + \mu_+ = 2p(X)$$

where μ_0 resp. μ_+ denotes the dimension of the radical resp. a maximal positive definite subspace of $H_2(X_t, \mathbb{R})$ (equipped with the intersection pairing). This formula is in this generality due to Steenbrink [12]; our proof is essentially that of Wahl [14] if we grant the globalization property of the appendix. If X is Gorenstein, then X_t admits a global nowhere zero holomorphic 2-form, so that $c_1(X_t) = 0$. Then (4.1b) gives us a formula for $\chi(X_t)$, which is also due to Steenbrink [12]

4.1f. $$\chi(X_t) = \chi(\tilde{X}) + c_1(\tilde{X})^2 + 12p(X)$$

For $n = 2m \geq 4$, all I can say is that $T_n[X_t]$ is a nontrivial linear combination of $\chi(X_t)$ and $c_m^2(X_t)$. It would be nice to have a natural condition on X (generalizing the Gorenstein property for $n = 2$) which implies the vanishing of $c_m(X_t)$, so that for these singularities $\chi(X_t)$ is an invariant of X. One instance where $c_m^2(X_t) = 0$ is when X has embedding dimension $\leq 2n-1$. Then X_t embeds in an open subset of $\mathbb{C}^{2n-1}$. If ν_{X_t} denotes its normal bundle, then $c(X_t) = c(\nu_{X_t})^{-1}$ (equality of total Chern classes), so that $c(X_t)$ is in the subalgebra of $H^*(X_t)$ generated by elements of degree $< n$. From this it follows that $c_m^2(X_t) = 0$.

For $n = 2m+1$ odd ≥ 3,

4.1g. $$T_n[\tilde{X}] = p(X)$$

imposes a nontrivial condition on X, apparently necessary in order that X admits a smoothing. The following example may illustrate this.

Example 4.2. Let E be a nonsingular projective variety of even dimension $2m$ and let l be an ample line bundle on E. Then

$$X := \operatorname{Spec} \bigoplus_{k=1}^{\infty} H^0(E, l^k)$$

is a quasi-homogeneous cone of dimension $n := 2m+1$ which is smooth outside its vertex. If $\tilde{X}$ denotes the total space of l^{-1}, then the natural map $\pi: \tilde{X} \to X$ is a resolution of X whose exceptional locus is just the zero section of $\tilde{X}$. If we identify the zero section with E, then its normal bundle in X equals l^{-1}.

We first express $T_n[\tilde{X}]$ in terms of the Chern classes of E and l. Put $\eta := c_1(l) \in H^2(E)$ and let $r: \tilde{X} \to E$ denote the projection. Consider the commutative diagram

$$\begin{tikzcd} H_c^{2m+k+2}(\tilde{X}) \arrow[r, "j^*"] & H^{2m+k+2}(\tilde{X}) \\ H^{2m+k}(E) \arrow[u, "\simeq"', "\gamma"] \arrow[r, "\cup(-\eta)"'] & H^{2m+k+2}(E) \arrow[u, "\simeq"', "r^*"] \end{tikzcd}$$

where γ is the Thom-isomorphism. For $k \geq 0$, the bottom map is surjective by the hard Lefschetz theorem. This enables us to relate the intersection calculus on $\tilde{X}$ to that on E. For instance, if $u \in H^2(E)$ and $v \in H^{4m}(E)$, choose $\tilde{v} \in H^{4m-2}(E)$ such that $\tilde{v} \cup (-\eta) = v$, e.g. $v = -(\deg l)^{-1} v[E]\eta^{2m-1}$, and then

4.2a. $$\begin{aligned}(r^*(u) \cdot r^*(v))[\tilde{X}] &= (r^*(u) \cup \gamma(\tilde{v}))[\tilde{X}] = (u \cup \tilde{v})[\tilde{X}] \\ &= (u \cup \tilde{v})[E] = -(\deg l)^{-1} \cdot (\eta^{2m-1} \cdot u) \cdot v[E].\end{aligned}$$

Because of the exact sequence

$$0 \to r^*\eta^{-1} \to \tau_{\tilde{X}} + r^*\tau_E \to 0$$

we have $T(\tau_{\tilde{X}}) = r^*T(l^{-1})\cdot r^*T(\tau_E)$ (multiplicativity of the total Todd class.) Substituting

$$T(l^{-1}) = 1 - \tfrac{1}{2}\eta + \Sigma_{k\geq 1}(-1)^{k+1}\frac{B_k}{(2k)!}\eta^{2k}$$

(where B_k denotes the kth Bernoulli number), gives

4.2b. $$T_n[\tilde{X}] = r^*T_n(\tau_E)[\tilde{X}] - \tfrac{1}{2}\eta T_{2m}(\tau_E)[\tilde{X}] + \Sigma_{k=1}^{m}(-1)^{k+1}\frac{B_k}{(2k)!}(\eta^{2k}\cdot T_{2m-2k-1})[\tilde{X}]$$

Since n is odd, T_n is divisible by $c_1: T_n = c_1 T'_{2m}$ with $T'_{2m}\in\mathbb{Q}[c_1,\ldots,c_{2m}]$. This enables us to apply (4.2a) to each term of right hand side of (4.2b):

$$T_n[\tilde{X}] = \frac{-1}{\deg(l)}(\eta^{2m-1}\cdot c_1(T_E))T'_{2m}[E]$$

$$+\tfrac{1}{2}T_{2m}[E] + \Sigma_{k=1}^{m}(-1)^k\frac{B_k}{(2k)!}(\eta^{2k-1}\cdot T_{2m-2k+1}(\tau_E))[E].$$

On the other hand,

$$p(X) = \Sigma_{i\geq 1}(-1)^i \dim_{\mathbb{C}} \bigoplus_{k\geq 0} H^i(E, l^k) = \Sigma_{i\geq 1, k\geq 0}(-1)^i h^i(E, l^k).$$

To see that this is in general different from $T_n[\tilde{X}]$, let us assume that $m = 1$, the canonical bundle K_E of E is very ample and $l = K_E$. Then $h^i(E, l^k) = 0$ for $i \geq 1$, $k \geq 2$, by the Kodaira vanishing theorem and hence

$$p(X) = 2\chi(\mathcal{O}_E) - 1 - p_g(E) = \chi(\mathcal{O}_E) - q(E).$$

Since $T_3 = 1/24\, c_1c_2$, we have $T'_2 = 1/24 c_2$. Furthermore, $\eta = -c_1$. So

$$T_3[\tilde{X}] = \frac{c_2}{24} + \frac{(c_1^2+c_2)}{24} + \frac{c_1^2}{24} = \chi(\mathcal{O}_E).$$

Hence $p(X) = T_3[\tilde{X}]$ implies $q(E) = 0$, which is clearly a nontrivial condition. Another class of examples is obtained by assuming that K_E is trivial, η is very ample, and $m \geq 1$ arbitrary: then $p(X) = \chi(\mathcal{O}_E) - 1$ by the Kodaira vanishing theorem, while $T_n[\tilde{X}] = \tfrac{1}{2}T_{2m} = \tfrac{1}{2}\chi(\mathcal{O}_E)$, so that X smoothable implies $\chi(\mathcal{O}_E) = 2$. In particular, E is not an abelian variety.

4.3. We now assume that X is normal of dimension $n \geq 2$, and we choose θ as our natural sheaf. There is a natural injection $\pi_*\theta_{\tilde{X}} \to \theta_X$. If we choose the resolution $\tilde{X} \to X$ equivariant in the sense of Hironaka [6], then this injection is in fact an isomorphism. We shall assume that this is the case. Since $\theta_{\mathcal{X}/\Delta}$ has depth ≥ 2, the natural map $\theta_{\mathcal{X}/\Delta}\otimes\mathcal{O}_X \to \theta_X$ is injective. If we denote the dimension of its cokernel by $\beta(\mathcal{X}/\Delta)$, then the θ-index of $X \to \mathcal{X} \to \Delta$ equals $-\beta(\mathcal{X}/\Delta)$. The number $\beta(\mathcal{X}/\Delta)$ equals the dimension of the smoothing component of a miniversal deformation of X on which the smoothing $\mathcal{X} \to \Delta$ takes place. This was conjectured by Wahl and has been recently proved by Greuel and the author [4]. So (3.3) gives:

4.3a. $$\dim S + P_\theta[X_t] = -\Sigma_{i\geq 1}(-1)^i \dim_{\mathbb{C}} H^i(\theta_{\tilde{X}}) + P_\theta[\tilde{X}]$$

Write $P_\theta[X_t] = \gamma_n\chi(X_t) + \delta_n c_{n/2}^2(X_t)$. For odd n, these coefficients are easily computed: $\delta_n = 0$, $\gamma_n = 1/(n-1)!$ Thus we find:

4.4. If n is odd, then for any smoothing component S of universal deformation of X,

$$\dim S + \frac{1}{(n-1)!}\chi(\text{general fibre over } S) = -\Sigma_{i\geq 1}(-1)^i \dim_{\mathbb{C}} H^i(\theta_{\tilde{X}}) + P_\theta[\tilde{X}]$$

in particular, the left hand side is independent of S!

If $n = 2m$ is even, we can eliminate $c_m^2(X_t)$ from (4.1b) and (4.3a) and thus get a formula similar to (4.4). For $n = 2$, we thus find Wahl's formula [14], (4.6.1).

4.5. We finally observe that for any natural sheaf $\mathscr{F}$ with $\mathscr{F}_{\mathscr{X}/\Delta,x}$ torsion free and $\mathscr{F}_{\mathscr{X}/\Delta}\otimes\mathcal{O}_{X,x} \cong \mathscr{F}_{X,x}$ the $\mathscr{F}$-index does not depend on the smoothing and hence gives rise to a $\mathbb{Q}$-linear combination of $\chi(X_t)$ and $c_{1/2n}^2(X_t)$ which is expressible in terms of X and its resolution. So for odd n this yields a formula for $\chi(X_t)$ if the coefficient of c_n in $P_{\mathscr{F}}$ is nonzero; for even n we get a formula for $\chi(X_t)$ if the relation is independent of (4.1b). For instance, if $\Omega_{\mathscr{X}/\Delta,x}$ has no torsion, then (3.3) gives

4.4a. $$P_{\Omega^1}[X_t] = P_{\Omega^1}[\tilde{X}] - \dim_{\mathbb{C}} \pi_*\Omega^1_{\tilde{X}}/\Omega^1_X - \Sigma_{i\geq 1}(-1)^i \dim_{\mathbb{C}} H^i(\Omega^1_{\tilde{X}}).$$

For odd n, the left hand side of this equation is $-(1/(n-1)!)\chi(X_t)$.

APPENDIX: GLOBALIZING SMOOTHINGS

The purpose of this appendix is to prove that Wahl's globalization hypothesis is always verified:

THEOREM. *Let $f\colon (\mathscr{X}, x) \to (\mathbb{C}, 0)$ be a smoothing over $(\mathbb{C}, 0)$ of an isolated singularity. Then there is a flat projective morphism $F\colon \mathscr{Z} \to \mathbb{C}$, a point $z \in Z_0$ and an isomorphism $h\colon (\mathscr{X}, x) \to (\mathscr{Z}, z)$ such that $F \circ h = f$ and F is smooth along $Z_0 - \{z\}$.*

For the proof we need a slight generalization of a result of Mather on right-equivalence. Since $\mathscr{X}$ and f are smooth outside the closed point, the jacobi ideal $\langle \theta_{\mathscr{X},x}, df\rangle \subset \mathcal{O}_{\mathscr{X},x}$ defines $\{x\}$ and hence contains a power of $m_{\mathscr{X},x}$.

LEMMA. *If $k \in \mathbb{N}$ is such that $m^{k-2}_{\mathscr{X},x} \subset \langle \theta_{\mathscr{X},x}, df\rangle$, then for any $g \in \mathcal{O}_{\mathscr{X},x}$ with $g - f \in m^k_{\mathscr{X},x}$ there exists an automorphism h of $(\mathscr{X}, x)$ such that $g = f \circ h$.*

Once it is observed that for an arbitrary analytic germ (Y, y) (whether reduced or not), any $\xi \in \theta_{Y,y}$ generates an analytic local one-parameter group of automorphisms (given as a germ $(\mathbb{C} \times Y, 0 \times y) \to (Y, y)$), the proof is literally the same as Mather's in the case where Y, y is smooth ([15], Thm. (2.9)). We therefore omit it.

Proof of the theorem. The point x is (at most) an isolated singular point of $\mathscr{X}$, so following Artin ([1], 3.8) there is an affine algebraic variety Y and point $y \in Y$ such that (Y, y) and $(\mathscr{X}, x)$ have isomorphic formal completions. According to Hironaka [5] this implies that (Y, y) and $(\mathscr{X}, x)$ are analytically isomorphic. Embed Y in some affine space $\mathbb{C}^N$ such that $y = 0$ and let $\bar{Y} \subset \mathbb{P}^N$ be its projective completion. By resolving the singularities of $Y - \{y\}$ we can suppose that $\bar{Y} - \{0\}$ is nonsingular and by a small perturbation of the hyperplane at infinity $\mathbb{P}^{N-1}_\infty$, we may in addition assume that $\mathbb{P}^{N-1}_\infty$ meets $\bar{Y}$ transversally, so that $\bar{Y}_\infty := \bar{Y} \cap \mathbb{P}^{N-1}_\infty$ is also nonsingular. Choose a polynomial $\tilde{g} \in \mathbb{C}[z_1, \ldots, z_N]$ of degree k (with k as in the lemma) whose restriction g to Y is such that

(i) $g - f \in m^k_{Y,0}$ and

(ii) the degree k-part of $\tilde{g}$ defines a nonsingular projective hypersurface $H(\tilde{g})$ in $\mathbb{P}^{N-1}_\infty$ which meets $\bar{Y}_\infty$ transversally.

If $N(g)$ denotes the zero-divisor of g on $\bar{Y}$, then $N(g) \cap \bar{Y}_\infty = H(\tilde{g}) \cap \bar{Y}_\infty$ so that this intersection is nonsingular by (ii). Now the singular locus of $N(g)$ is contained in the affine variety Y and hence finite. Choose $\phi \in \mathbb{C}[z_1, \ldots, z_N]$ homogeneous of degree k and nonzero in all the (finitely) many singular points of $N(g) - \{0\}$. If we replace $\tilde{g}$ by $\tilde{g} + t\phi$ ($t \in \mathbb{C} - 0$), then (i) still holds, (ii) continues to hold if t is sufficiently small and

(iii) $N(g) - \{0\}$ is nonsingular for generic t by Bertini's theorem. We may therefore assume that (iii) also holds. Let $\mathscr{Z} \subset \bar{Y} \times \mathbb{C}$ be the closure of the graph of g. Note that $\mathscr{Z}$ can be identified with the blow-up of $\bar{Y}$ along $N(g) \cap \bar{Y}_\infty$ with the strict transform of $\bar{Y}_\infty$ removed, so $\mathscr{Z}$ is nonsingular. Letting $F\colon \mathscr{Z} \to \mathbb{C}$ be the projection on the second factor, then F is a projective flat morphism, $F^{-1}(0) - \{0, 0\} \simeq N(g)$

$-\{0\}$ is nonsingular of multiplicity 1 (so F is smooth along $F^{-1}(0)-\{(0,0)\}$) and finally, the germ of F at $(0,0)$ is right-equivalent to f, by the lemma. This proves the theorem.

Remark. As already mentioned in the introduction, the globalization property has implications for Wahl's paper [14]. His Thm. (3.13) now holds for any smoothing of a normal surface singularity; and if we also invoke the main result of [4], we see that the same is true for Cor. (4.6), while Thm. (4.10) is now valid for any smoothing of a Gorenstein surface singularity. If we take into account the work of Steenbrink [12] and Greuel–Steenbrink (*Proc. Symp. Pure Math.* **40,** Arcata (1983), Part 1, 535–545), it follows that all the conjectures made in [14] have been answered affirmatively.

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